ON A STABILITY CRITERION FOR CREEP

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S. A. SHESTERIKOV
(Moscow)

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Stability in the presence of creep has been studied by different authors; some of these investigations were made as much as ten years ago.

The majority of the published papers are concerned with problems of the stability of longitudinally compressed rods, since this represents the simplest formulation of the problem by which many of the particular characteristics of stability can be elucidated.

It has to be noted that there exists a series of principally different formulations of the problems of stability in the presence of creep. In this paper, consideration will be restricted to stability of rectilinear forms of equilibrium.

The first study of stability in the presence of creep is due to Rzhanitsyn [1]. This paper considers the stability of a rod the material of which obeys a law of the form

$$n\sigma^{\cdot} + \sigma = H\varepsilon + En\varepsilon^{\cdot} \tag{1}$$

This equation describes limited creep, i.e. for constant loading the displacements approach a definite limit. It must be emphasized that the behavior of a rod for longitudinal bending will differ essentially depending on the way in which the creep of the material depends on the magnitudes of the accumulated deformation. In what follows, the limiting expression in this sense will be stated. For the case of steady creep, it has been shown [2] that for any eccentricity, however arbitrary and small, the rod moves with monotonically growing velocity amplitude; therefore, unsteady creep will be considered below.

Now consider the stability of a rod whose material obeys the relation (1). Rzhanitsyn in his paper in 1946, as also in subsequent work, considered straight rods, compressed by central forces, and, constructing the equilibrium equations for the disturbed state (without defining sufficiently exactly the disturbance under consideration) he obtained the quasi-static equations of motion. Thus, as has been noted by the same author [3], in this work the initial conditions were not fulfilled. Further, he obtained an expression for the critical load which was determined by the Euler formula, where E was replaced by the modulus of elongation H. This result does not give rise to any doubt, but the method of its deduction produces some consternation, since once definite hypotheses regarding the dependence for the material have been introduced, it is natural to expect in the end to obtain the correct solution of the problem. Therefore, it will be interesting to proceed to the solution of the same problem with a somewhat different attitude.

It is known that for the solution of problems of elastic stability of rods in a linear formulation there exist several methods among which the most popular ones are the following: the existence for given loads of two states of equilibrium – the stability of the rod in vibrations and the boundedness of deflections when the rod has initial curvature. In a given case of creep there applies the circumstance that for constant loading, in general, the state of deformation does not remain constant. Therefore, the question regarding the possibility of the simultaneous existence of two states of equilibrium is determined by the instantaneous characteristics of the body, and, since in the majority of theories it is assumed that the instantaneous characteristics are purely elastic, then, besides the elastic loss of stability, nothing else in this direction can be obtained. In fact, it is for this reason that all attempts to solve the problem by exact methods have encountered such a contradiction as the impossibility of satisfying the initial conditions. Consequently, such a method may not give the fundamental results completely.

The second method of investigation, based on the stability of the vibrations, is used for the determination of the critical state; it has been used in a number of papers (for example, in [4]). It must be noted that this method, generally speaking, requires an extension of the introduced hypotheses regarding the behavior of the material during the process of vibration. However, this may be avoided, particularly since for the velocities of creep inertia does not play an important role.

Therefore, consider the behavior of a hinged rod which is compressed by longitudinal forces and the material of which obeys the relation (1); it will be assumed that the rod has the initial deflections

$$y = a_{00} \sin \frac{\pi x}{L} \tag{2}$$

Using (1) one may express σ in terms of the strain ϵ; if the hypothesis of plane sections is introduced, and if one considers everything in the linear formulation, then σ may be expressed in terms of the second derivative of the deflection. Substituting the obtained expression for σ in the equilibrium equation one finally obtains the equation for the amplitude of the deflection a for longitudinal bending:

$$E(P_e - P)a + (P_e H - PE)\frac{a}{n} = P_e \frac{H}{n} a_{00} \qquad \left(P_e = \frac{\pi^2 EI}{L^2}\right) \tag{3}$$

Here P_e is the Euler critical force,

$$a(0) = \frac{P}{P_e - P} a_{00} \quad \text{for} \quad t = 0$$

Therefore the solution of (3) has the form

$$\frac{a}{a_{00}} = \frac{P_e H}{P_e H - PE}\left[1 - \exp\left(-\frac{P_e H - PE}{P_e - P}\frac{t}{nE}\right)\right] + \frac{P}{P_e - P}\exp\left(-\frac{P_e H - PE}{P_e - P}\frac{t}{nE}\right) \tag{4}$$

for the condition $P_e H - PE \neq 0$; obviously, for $PE > P_e H$, the deflection becomes unbounded with time, while for $PE < P_e H$ it tends to the limit

$$a = \frac{P_e}{P_e - PE\;H} a_{00} \tag{5}$$

In the case $P_e H = PE$, the solution of (3) has the form

$$\frac{a}{a_{00}} = \frac{P}{P_e - P} + \frac{P_e(H/E)}{n(P_e - P)} t \tag{6}$$

which likewise leads to $a \to \infty$ for $t \to \infty$. The deduction from the solution obtained coincides with that of Rzhanitsyn regarding the critical stress, but it is free from certain deficiencies.

It must be underlined that in distinction from the studies in the elastic region for creep, it is possible to obtain critical values for short term excessive loading, since an essential role is played here by the time history and not the instantaneous characteristics. Of course, the short term loading may not exceed the elastic critical force.

Consider next the arbitrary nonlinear hardening law of the type

$$p = f(p, \sigma) \tag{7}$$

where p is the plastic deformation. In [4] the problem of the stability of a compressed rod has been studied for a creep law of the form (7). In this paper, there were proposed quasistatic and dynamic approaches to the solution of the problem and it was shown that they give identical results for the determination of the stability of a state. Just as in the work of Rzhanitsyn, for the quasistatic method the initial conditions are not fulfilled. Here, successively pursuing the above procedure, it will be shown that the analogous results may be obtained without any internal contradictions in the deduction. In addition, the method of investigation of longitudinal bending in the presence of initial deformation is free from the contradiction which occurs in the quasistatic study, that with time the initial deformation of the compressed bar may decrease, which does not correspond to logical reasoning nor to experimental results.

Before proceeding to the study, one circumstance must be noted. For creep of the elements of a structure, for example in pure bending, a constant growth of the deflections with time is observed; therefore, the definition of stability of a structure as the boundedness of deflections for unlimited time is inadmissible for practical purposes. Hence another definition of the stability of a longitudinally compressed bar will be introduced.

The state of a longitudinally compressed bar will be assumed to be stable if for constant load its initial deflection does not increase faster than a linear function of time; or, in other words, the rate of growth of the deflections with time does not increase and the state is assumed to be unstable if the deflections grow with increasing rate.

A very particular definition of stability has been given here which is in need of a refinement and extension. The process of longitudinal bending of a rod in the case of creep can be broken up into two stages: the first when the increase of the deflections is bounded (this applies for bounded creep) or takes place with approximately constant rate, and the second when the increase of the deflections bears an avalanche-type character (the increase of the deflection growing with the rate of deflection).

This definition also has a definite meaning for the explanation of experimental results in which first there is observed a convergence of the cross-beams of the testing machine with constant velocity, and then a rapid growth in this velocity.

In accordance with the introduced definition, one and the same rod for one and the same loading may find itself in a stable or in an unstable state depending on the accumulated plastic deformation.

Consider a hinged bar, compressed by a longitudinal force P; let the initial deflection be given by (2). It will be assumed that the deflections are small and therefore for an increase of the stresses and deformations, characterizing the bending of the rod, one can take the variational relation (7):

$$\delta p^{\cdot} = \frac{\partial f}{\partial p}\delta p + \frac{\partial f}{\partial \sigma}\delta\sigma = \lambda\delta p + \mu\delta\sigma \qquad (8)$$

Taking into consideration that

$$\delta\varepsilon = \varkappa z - \varkappa_0 z = \delta p + \frac{\delta\sigma}{E}, \qquad \int_F \delta\sigma z dz = -P^{\cdot}{}_y$$

and letting $y = a \sin(\pi x/L)$, for a one obtains the equation

$$(P_e - P)\, a^{\cdot} + (-\lambda P_e + \lambda P - E\mu P)\, a = -P_e \lambda a_{00} + \frac{aP^{\cdot}}{E} \tag{9}$$

Introducing the non-dimensional deflection

$$u = \frac{a}{a_0} = \frac{a}{a_{00}}\left(1 - \frac{P}{P_e}\right) \tag{10}$$

and assuming $P^{\cdot} = 0$, equation (9) becomes

$$u^{\cdot} + u\,(-\lambda - E\beta\mu) = -\lambda \qquad \left(\beta = \frac{P}{P_e - P}\right) \tag{11}$$

Equation (11) does not depend on the initial deflection a_{00}, the initial condition having the form: $u(0) = 1$. Hence, all qualitative deductions regarding the determination of stable and unstable states of the rod will not depend on the magnitude of the initial deflections of the rod.

Equation (11) will now be studied. Let equation (7) characterize hardening when $\lambda < 0$, $\mu > 0$. For $t = 0$ one has $u > 0$, i.e. after the loading has been imposed the deflections grow. Now study the increase of the deflection with time. Since μ and λ are the characteristic quantities of the basic deformed state of the rod, obtained by integration of (7) for σ = const, they will themselves represent known functions of time. Thus one obtains readily the solution of equation (11) which takes the form

$$u = \left\{c_1 - \int_0^t \lambda \exp\left[-\int_0^\tau (E\mu\beta + \lambda)\, d\tau\right] d\tau\right\} \exp\left[\int_0^\tau (E\mu\beta + \lambda)\, d\tau\right] \tag{12}$$

From (12) one may find the regions in which u does not grow and in which it grows, which in accordance with the introduced definition gives the regions of stability and instability. It is hardly possible to study this relation in its general form, but certain results may be obtained by consideration of equation (11). If it is assumed that the coefficients in this equation vary little and that they may be taken to be constant, then the solution of (11) may be written in the form

$$u = \frac{\lambda}{\lambda + E\mu\beta} + ce^{(-\lambda - E\beta\mu)\, t} \tag{13}$$

Let $\lambda + E\mu\beta < 0$ and $\lambda < 0$; since the deflections increase with increasing time t, then $c < 0$ and, consequently, $u < 0$; therefore u does not increase and one has a stable state.

If $\lambda + E\mu\beta > 0$ and $\lambda < 0$, then $c > 0$; in this case one has an unstable state.

The boundary between stability and instability will be the condition $\lambda + E\mu\beta = 0$ which coincides exactly with the quasistatic criterion of stability.

It is of interest here to consider the one case when the equality $\lambda + E\mu\beta = 0$ is satisfied at all times except at one instant; consequently, for any p, assuming in addition that f may be represented in the form $f = g(p)\psi(\sigma)$, one obtains

$$\psi\frac{dg}{dp} + E\beta g\frac{d\psi}{d\sigma} = 0 \tag{14}$$

and, since $\sigma = \text{const}$, one has $g = e^{-\gamma p}$

$$-\psi\gamma + E\beta\frac{d\psi}{d\sigma} = 0 \tag{15}$$

The second relation determines the critical value σ_* and it is readily shown that for $\sigma < \sigma_*$ the rod is always stable, while for $\sigma > \sigma_*$ it is always unstable. The creep law of the form (15) describes unbounded creep, but here the creep deformation grows for constant stress like log t. It is clear that here the conditions are the same as for the relation (1) and it may be said that this law represents the upper bound for relations, describing bounded and, as shown above, even unbounded creep, for which there exists a critical load which does not depend on the magnitude of the accumulated deformation. It should be stated that in the case of steady creep $\lambda = 0$ and, therefore, an unstable state will always prevail.

Equation (11) will now be studied with a view to changes in λ and μ for one particular creep law when the relation (7) has the form

$$p^{\cdot} = f(p, \sigma) = \varphi(p)\psi(\sigma) \tag{16}$$

In accordance with this equality

$$dt = \frac{dp}{f} \tag{17}$$

From (11) one obtains

$$\frac{du}{dp} + u\left(-\frac{d\ln\varphi}{dp} - \frac{n}{\varepsilon_e - \varepsilon}\right) = -\frac{d\ln\varphi}{dp} \qquad \left(n = \sigma\frac{d\ln\psi}{d\sigma},\ \varepsilon = \frac{\sigma}{E}\right) \tag{18}$$

Here ϵ_e is the Euler critical deformation for the rod, p as a function of time is determined by relation (17). Integrating (18), taking into consideration the initial condition $u = 1$ for $t = 0$, and this means also for $p = 0$, one finds

$$u = \varphi e^{p_1} \int_0^p e^{-p_1} d\left(\frac{1}{\varphi}\right), \qquad p_1 = \frac{np}{\varepsilon_e - \varepsilon} \tag{19}$$

The condition of the critical state has the form

$$\frac{d^2u}{dt^2} = 0 \tag{20}$$

In its general form, the integral (19) cannot be expressed in terms of elementary functions. A particular form of the law (16) will now be studied. Let

$$\varphi(p) = p^{-\alpha} \tag{21}$$

Then (19) has the form

$$u = p^{-\alpha} e^{p_1} \int_0^p e^{-p_1}\, dp^{\alpha} \tag{22}$$

In this expression, when a is an integer, the integration is easily performed; as in the case $a = 1$, the equality (22) has the form

$$u = \frac{e^{p_1} - 1}{p_1} \tag{23}$$

Using (20), for the determination of the critical state one obtains the condition

$$e^{p_1} = \frac{1}{1 - p_1 + {}^1/_3\, p_1^2} \tag{24}$$

In addition to the solution $p_1 = 0$, there still exists the solution

$$p_1 = 1.36 \tag{25}$$

By the approximate and quasistatic criteria one has the condition

$$p_1 = \alpha \tag{26}$$

Condition (25), being more exact and justified, gives larger values of the critical time than the approximate theory, as, generally speaking, corresponds to experimental evidence.

In the cases $a = 2$ and $a = 3$ one obtains by an analogous method

$$p_1 = 3.10 \quad (\alpha = 2), \qquad p_1 = 4.98 \quad (\alpha = 3) \tag{27}$$

It is obvious that for increasing a the difference between the exact and the approximate methods increases.

All the statements are correct when sufficiently smooth relations of the form (8) are under consideration, i.e. such relations that the expression $-\lambda - \beta\mu E$ changes its sign only once. Otherwise there may exist several zones of stability and instability.

In conclusion, it must be said that usually the linear formulation gives upper values of the critical forces. In the present case one has as a lower estimate of the carrying capacity of the bar, i.e. at the attainment of the critical state the rod does not fail, but there will be observed only an accelerated increase of the deflections.

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